Problem Set I: Due Monday, January 25, 2010

1.) Kulsrud, Chapter 3, Problem 1

2.) Kulsrud, Chapter 3, Problem 2

3.) Kulsrud, Chapter 3, Problem 3

4.) *Electron MHD* (EMHD)

This extended problem introduces you to EMHD and challenges you to apply what you've learned about MHD to understand the structures of a different system of fluid equations. In EMHD, the ions are stationary and the "fluid" is a fluid of electrons. EMHD is useful in problems involving fast Z-pinches, filamentation and magnetic field generation in laser plasmas, Fast Igniter, etc.

The basic equations of EMHD are the electron momentum balance equation

(1) $$\frac{\partial \underline{\mathrm{v}}}{\partial t} + \underline{\mathrm{v}} \cdot \underline{\nabla} \underline{\mathrm{v}} = -\frac{q}{m}\underline{E} - \frac{\nabla P}{\rho} - \frac{q}{mc}\left(\underline{\mathrm{v}} \times \underline{B}\right) - \nu \underline{\mathrm{v}},$$

(2) $$\underline{J} = -nq\underline{\mathrm{v}},$$

and continuity

(3) $$\underline{\nabla} \cdot \underline{J} = 0.$$

Note that here, Ampere's law forces incompressibility of the mass flow $\rho \underline{\mathrm{v}}$. Here $\underline{\mathrm{v}}$ is the electron fluid velocity, ν is the electron-ion collision frequency, $q = |e|$, $m = m_e$. Of course, Maxwell's equations apply, but the displacement current is neglected.

i.) *Freezing-in*

Determine the freezing-in law for EMHD by taking the curl of Eqn. (1) and using the identity $\underline{\mathrm{v}} \cdot \underline{\nabla} \underline{\mathrm{v}} = \underline{\mathrm{v}} \times \underline{\omega} - \underline{\nabla}\left(\mathrm{v}^2/2\right)$.

Assume the electrons have $p = p(\rho)$. Approach this problem by trying to derive an equation for "something" which has the structure of the induction equation in MHD. Discuss the physics - what is the "something" and what is it frozen into? In retrospect, why is the frozen-in quantity obvious? How is freezing-in broken?

ii.) *Large Scale Limit*

Show that for $\ell^2 >> c^2/\omega_{pe}^2$, the dynamical equations for EMHD reduce to

$$\frac{\partial B}{\partial t} + \underline{\nabla} \times \left(\frac{\underline{J}}{nq} \times \underline{B} \right) = -\nu \underline{\nabla} \times \left(\frac{\underline{J}}{nq} \right)$$

$$\underline{\nabla} \cdot \underline{J} = 0 ; \quad \underline{\nabla} \cdot \underline{B} = 0 .$$

a.) Show that density remains constant here.

b.) Formulate an energy theorem for EMHD in this limit, by considering the energy content of a "blob" of EMHD fluid.

c.) Discuss the frozen-in law in this limit.

d.) Consider the case of a field $\underline{B} = B(x)\hat{z}$ and $n = n(y)$. Derive a general equation for a field with no tension, and specialize it to the case considered. You may neglect collisions. Prove that (in the general case), magnetic flux is conserved.

e.) Retaining a constant resistivity, solve the resulting equation (from part d.) for B(x) *exactly*, by applying the Hopf-Cole transformation from Burgers' Equation. [N.B.: Whitham, Chapter 4, is a good reference on Burgers' Equation.]

5.) Kulsrud, Chapter 4, Problem 2

6.) Consider a spectrum of magnetic fluctuations in a periodic cylinder model of a tokamak. Assume the toroidal field is not perturbed.

a.) Show that periodicity requires:

$$\hat{\underline{B}} = \sum_{m,n} \underline{B}_{m,n}(r) e^{i(m\theta - n\phi)} e^{-i\omega_{m,n}t} .$$

b.) Assuming particles can only move along fluctuating field lines, show that the kinetic equation becomes:

$$\frac{\partial f}{\partial t} + v_{\|} \frac{\underline{B}}{B_0} \cdot \underline{\nabla} f = 0 .$$

Here, take $\underline{B}_0 = B_T \hat{z} + B_\theta(r)\hat{\theta}$, with $|B_T| >> |B_\theta|$ and $|\hat{\underline{B}}| << |B_0|$.

c.) Use quasilinear theory to calculate the *radial* flux of particles. Under what circumstances is quasi-linear theory valid? Show for stationary perturbations:

$$D = v_{\|}^2 \sum_{m,n} \frac{\left|\tilde{B}^{(r)}_{r\,m,n}\right|^2}{B_0^2} \pi\delta\left(\omega_{m,n} - k_{\|} v_{\|}\right),$$

where $k_{\|} = \underline{k} \cdot \underline{B}_0 / |\underline{B}_0|$.

d,) What happens for a spectrum of random static perturbations $(\omega_{m,n} \to 0)$? Calculate D and describe the condition necessary for validity of QLT then? Show $D = |v_{\|}| D_m$ and interpret the meaning of D_m.

e.) Assuming we are discussing electrons, use Ampere's Law to estimate the loss rate of electrons, assuming ions have a small current $\tilde{J}_{\|,i}$. Compare this with the test particle result above.

7.) Prove the energy conservation relation for MHD, as given by Kulsrud in Section 4.5. Show <u>ALL</u> steps clearly. Trace the cancellation of terms in the proof.

8a.) Show that $\underline{\omega}/\rho$ is frozen into an inviscid fluid. Here $\underline{\omega} = \underline{\nabla} \times \underline{V}$. Take $P = P(\rho)$.

b.) Prove Kelvin's Theorem for an inviscid fluid with $P = P(\rho)$.

c.) Compare and contrast the freezing-in laws and Theorems for MHD and a fluid with $P = P(\rho)$.